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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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**All contributions should be written legibly on one side only of the paper, and all diagrams should be neatly and accurately drawn on separate slips.**

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# THE INDIAN MATHEMATICAL SOCIETY.

## ANNOUNCEMENT.

The Committee are glad to announce that it has been decided to hold the next Conference of the Society at Nagpur on 24th, 25th, and 26th December 1928; and they request the co-operation of all members in this behalf.

Authors of papers to be presented to the Conference are requested to give the titles of their papers, with a brief summary, before the end of September 1928; and to send the papers themselves before the end of October 1928. These should be sent to the Hon. Editor, Dr. R. Vaidyanathaswami, M.A., D.Sc., F.R.S.E., Reader in Mathematics, University of Madras, Madras.

G. S. CHOWLA,
*Hon. Joint Secretary.*

**7.** We shall now study the quadrics in greater detail considering separately the cases where $\Gamma$ is an ellipse, hyperbola, parabola, or an imaginary conic.

*Suppose $\Gamma$ is an ellipse.* Of the two focal conics one is real and is a hyperbola while the other is imaginary. Denote the planes of these by $\pi$ and $\pi'$. If $l$ be any line in $\pi'$ the quadric $Q_l$ has its principal foci imaginary as these are the intersections of $l$ with the focal conic of the plane. The quadric, being real, must be either a hyperboloid of one sheet or an oblate spheroid. Now when $l$ is the minor axis of $\Gamma$, which is a line in $\pi'$, the quadric is an oblate spheroid. We may deduce from this that all the quadrics corresponding to lines in $\pi'$ are oblate spheroids; for, if one of them, say $Q'_l$ be a hyperboloid of one sheet, then as $l$ moves continuously into the position $l'$ the quadric $Q_l$ undergoes a continuous deformation into the quadric $Q'_l$. But as one of them is an ellipsoid and the other a hyperboloid, the passage must be effected through an intermediary paraboloid*, and hence by (5), $l$ must take up a position in which it is parallel to one of the asymptotes of the focal conic in $\pi'$. But this is impossible since the directions of the asmptotes are imaginary. Hence

all the quadrics corresponding to lines in $\pi'$ are oblate spheroids. ... (6)

Consider next lines in the plane $\pi$. The focal conic is here a real hyperbola. If $l$ coincide with the major axis of $\Gamma$, the quadric $Q_l$ is a prolate spheroid. It can be shown by arguments similar to those used in the preceding paragraph that if a line cuts both the branches of the focal hyperbola, the corresponding quadric must be necessarily a prolate spheroid. For, in the first instance, since its principal foci are real it must be either a prolate spheroid or a hyperboloid of two sheets, and it cannot be the latter since the line $l$ can move from any position in which it cuts both branches into any other position fulfiling the same condition

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* The section by the plane at infinity is real in one case and imaginary in the other. The passage must be effected by a pair of straight lines which corresponds to a paraboloid.



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without becoming at any stage parallel to either asymptote of the focal conic, so that the quadric never becomes a paraboloid during the process.

We have already seen that when $l$ is a tangent to the focal conic the quadric becomes a cone. This cone is real (contains real generators) since it is obtained by the rotation of a real conic about a real line. If $l$ moves slightly the quadric of revolution is one which approximates to a real cone and is hence a hyperboloid of one or two sheets.* The former has its principal foci imaginary while for the latter these are real; hence these two cases correspond to positions of $l$ where it does not cut the focal hyperbola and the case where it meets the same branch in two real points. Summarising:

When $l$ lies in the plane $\pi$ and meets the focal hyperbola in two real points, the quadric $Q_l$ is a hyperboloid of two sheets or a prolate spheroid, according as these two points lie on the same or different branches; if the line $l$ meets the focal hyperbola in a pair of imaginary points the quadric of revolution is a hyperboloid of one sheet. ... ... ... ... (7)

There are two systems of paraboloids through $\Gamma$ whose axes are parallel to one or other of the asymptotes of the focal conic, and there are two right circular cylinders whose axes are actually the asymptotes.

8. *Let $\Gamma$ be a hyperbola.*

Here there is a focal ellipse which is real and lies in one principal plane $\pi$, while the focal conic in the other plane $\pi'$ is imaginary. A rotation about a line $l$ in $\pi'$ coincident with the conjugate axis of $\Gamma$ gives us a hyperboloid of one sheet, and by the principle of continuity we infer as before that all quadrics corresponding to lines in $\pi'$ are likewise hyperboloids of one sheet.

A line in $\pi$ which meets the focal ellipse corresponds to a hyperboloid of two sheets, and one which does not meet the ellipse to a hyper-

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* *Vide* in this connection the second foot-note on page 220.

boloid of one sheet, while tangents to the focal ellipse correspond to cones. There are no ellipsoids as may also be inferred also from the fact that no ellipsoid can give a hyperbolic section. There are no paraboloids and no cylinders of revolution.

**9.** *Let* $\Gamma$ *be a parabola.*

There is now only one principal plane $\pi$, and a focal narabola the tangents to which correspond to cones of revolution. When $l$ moves from its position of tangency, the core changes into a hyperboloid of one or two sheets according as $l$ in its new position does not meet the parabola or cuts it in real points. There are no ellipsoids, as is otherwise obvious, and but one system of paraboloids corresponding to lines which meet the parabola at one point at infinity, *i.e.*, to diameters of the focal conic.

**10.** The case where $\Gamma$ *is a circle* is a special one. The principal planes are indeterminate and there is a complex of lines $l$, namely, all those which meet the axis of the circle which can serve as axes of revolution for quadrics through $\Gamma$, but all the quadrics thus generated are spheres. In fact our method of investigation fails here, for while there exist quadrics of revolution through a circle, they cannot all be obtained by revolving the circle about the axis of the quadric since, except in the case of spheres, the axis of revolution of the quadric is also the axis of the circle.

If the conic $\Gamma$ be *a pair of straight lines* the focal ellipse shrinks into a point O, the intersection of the two lines. Any line lying in either of the principal planes and passing through the point O may be considered to be a tangent to the focal ellipse and corresponds to a cone. On the other hand, if the line $l$ in either of the principal planes does not pass through the point O, the quadric is a hyperboloid of one sheet. If $\Gamma$ consist of a *pair of parallel lines*, one of the principal planes is indeterminate and may be any plane perppendicular to the lines, while the other plane is definite. The quadrics are all hyperboloids of one sheet and, in particular, cylinders when the line $l$ is parallel to the given lines.

**11.** Lastly suppose $\Gamma$ is an *imaginary conic.** This case is of peculiar interest as the results here admit of a dual interpretation relating to spheres in Euclidean space developed in the next paragraph. Both the focal conics of $\Gamma$ are now real, one being an ellipse and the other a hyperbola. The ellipse meets the plane of $\Gamma$ in a pair of real points while the hyperbola meets it in imaginary points. Any tangent line to either of the focal conics corresponds to a cone of revolution as before, but the cone is now imaginary as otherwise its section by the real plane in which $\Gamma$ lies must be a real conic. Also an imaginary cone is a transitional form between a real ellipsoid and an imaginary quadric.† If $l$ is the line joining the intersections of the focal ellipse with the plane of $\Gamma$, the quadric of revolution corresponding to the axis $l$ is imaginary. Hence by the principle of continuity all the quadrics corresponding to lines which meet the focal ellipse in real points are imaginary quadrics, while those corresponding to lines which do not meet the focal ellipse are ellipsoids. The principal foci of these ellipsoids are imaginary

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* Throughout this investigation $\Gamma$ is taken to be a conic lying in a real plane and given by a real quadratic form equated to zero. If the form is positive definite the conic is called an *imaginary conic*. Similarly an imaginary quadric is one whose equation may be reduced to the form $ax^2+by^2+cz^2+dw^2=0$ where $a$, $b$, $c$ and $d$ are real and positive. When one of the co-efficients vanishes it is an imaginary cone. Every section of such a quadric or cone by a real plane is an 'imaginary conic.'

† As the theory of transitional forms plays such a leading part in this investigation, it may perhaps be worth while to indicate their analytical aspect. Suppose a particular position of $l$ gives the cone $ax^2+by^2+cz^2=0$; then a neighbouring position of $l$ will give a quadric whose equation reduced to its canonical form will be $(a+e_1)x^2+(b+e_2)y^2+(c+e_3)z^2=e_4$ where the $e$'s are real infinitesimals. If $a$, $b$, $c$ are all positive so that the cone is an imaginary one, the neighbouring quadric is an ellipsoid or an imaginary quadric according as $e_4$ is positive or negative. If the cone is real, two of the co-efficients may be taken positive and the third negative, and then the quadric is a hyperboloid of one or two sheets according as $e_4$ is positive or negative. Thus a real cone is a transitional form between a hyperboloid of one sheet and one of two sheets, while an imaginary cone is the transitional form between an ellipsoid and an imaginary quadric.

being their intersections with the focal ellipse ; hence all these ellipsoids of revolution are oblate spheroids.

Again when $\Gamma$ is rotated about its other principal axis (which lies in the plane of the focal hyperbola) the quadric of revolution is also imaginary, and hence by continuity the same is the case with regard to all lines $l$ which meet the focal hyperbola in a pair of imaginary points When such a line moves to a position of tangency, and later cuts the hyperbola (necessarily the same branch) in two real points, the quadric becomes an imaginary cone first, and later a real ellipsoid, since we have seen that an imaginary cone is a transitional form between an ellipsoid and an imaginary quadric. These ellipsoids of revolution are all prolate spheroids since the principal foci are real. If the line $l$ which meets the same branch twice, is to pass to a position where it meets both branches of the focal hyperbola, it must pass through a position of parallelism to one of the asymptotes, and the corresponding quadric of revolution is then a paraboloid. Also the paraboloid is a transitional form between an ellipsoid and a hyperboloid of two sheets Hence

> the quadric of revolution is imaginary when $l$ meets the focal ellipse or fails to meet the focal hyperbola. It is an ellipsoid when it fails to meet the focal ellipse or meets the same branch of the focal hyperbola twice, prolate in the former case and oblate in the latter. If the line meets both branches of the focal hyperbola the quadric is a hyperboloid of two sheets.

There is no hyperboloid of one sheet in the system as its section by the real plane in which $\Gamma$ lies cannot give an imaginary conic.

## 12. Isomorphic Systems.

It was stated in the introduction that the quadrics considered in this paper were those which passed through a conic $\Gamma$ and had double contact with another conic $\Gamma'$. If we take the latter of these as the Absolute which determines the metrical relations of a space $S'$, the quadrics considered so far are now 'spheres' which have double contact with a given conic $\Gamma'$. It is now possible to give a dual interpretation

of our results in relation to spheres in the space $S'$ by means of the following

TABLE OF EQUIVALENT TERMS.

| *Quadric system.* | *Sphere system.* |
|---|---|
| Quadric of revolution through a given conic $\Gamma$. | Sphere having double contact with a given conic $\Gamma'$. |
| Cones belonging to the system. | Point spheres belonging to the system. |
| Ellipsoid. | Sphere not meeting the plane of $\Gamma'$. |
| Hyperboloid of { one sheet. / two sheets. | Sphere intersecting the plane of $\Gamma'$ and having { real generators. / imaginary generators. |
| Paraboloid. | Sphere touching the plane of $\Gamma'$. |
| Axis of a quadric of revolution. | Polar line of chord of contact with respect to the sphere. |
| Principal focus. | Vertex of enveloping cone through $\Gamma'$. |
| Focal conics of $\Gamma$. | Focal conics of $\Gamma'$. |

The parallelism is particularly complete when the conic $\Gamma'$ is imaginary (§ 11) in which case the sphere system also relates to a Euclidean space with an imaginary Absolute. In this case there are no hyperboloids of one sheet in the system, and correlatively no spheres with real generating lines.

# FOCI IN COMPLEX GEOMETRY.

By N. Durairajan.

The object of this paper is to indicate the part played by the concept 'focus' in the geometry of the Gauss plane. The relation between focal properties and complex geometry is a reciprocal one. On the one hand, the method of complex representation may be utilised to elucidate the properties of real foci of curves; this is what is done in para. 1 of the paper, where a general theorem is stated, which gives immediately the position of the real foci of a curve given by a real areal tangential equation. On the other hand, the focus idea may itself be utilised as an instrument of geometrical interpretation in the invariant geometry of the complex plane; this is done in paras. 2 and 3, where pencils of binary forms (with real parameter), and the Jacobian of two binary forms are geometrically interpreted in the complex plane.

The last para. gives a treatment by complex geometry, of the Miquel point of a quadrangle (originally investigated by Clifford), namely, the point the feet of the perpendiculars from which, on the six sides of the quadrangle, lie on a conic. I obtain also the interesting result that five points can possess a Miquel point, only when they lie on a rectangular hyperbola.

## 1. Foci of curves in areal tangential equations.

Let $x$, $y$ be rectangular Cartesian co-ordinates, and let $l, m, n$ be taken as the co-ordinates of the line $lx + my + n = 0$. Then

THEOREM 1. The co-ordinates of the real foci of the envelope given by the real equation $\phi(l, m, n) = 0$ are determined by the roots of the equation $\phi(1, i, -z) = 0$, where $z = x + iy$.

To prove this, note that the equations to the circular lines through the real point $(x, y)$ are $X \pm iY - (x \pm iy) = 0$. If both these lines are tangent to the envelope, we must have

$$\phi(1, i, -(x + iy)) = 0 = \phi(1, -i, iy - x).$$

Since $\phi$ is a real quantic, either of these equations implies the other, so that the real foci are given by the roots of the equation $\phi(1, i, -z) = 0$,

Let now $z_1, z_2, \dots z_r$ be the complex numbers of $r$ real points $(x_1 y_1), (x_2 y_2) \dots (x_r y_r)$, so that $z_k = x_k + i y_k$. Also let $p_1 p_2 \dots p_r$ be the lengths of the perpendiculars (taken with the proper sign) from these points on the line $lx + my + n = 0$. Then

THEOREM 2. The real foci of the envelope given by the homogenous equation $\phi(p_1, p_2, \dots p_r) = 0$, are determined by the roots of the equation

$$\phi(z - z_1, z - z_2, \dots z - z_r) = 0.$$

For $p_k = (lx_k + my_k + n)/(l^2 + m^2)^{\frac{1}{2}}$. Substituting in $\phi(p_1, p_2, \dots p_r) = 0$. it follows from the homogeneity of $\phi$ that the equation to the envelope is

$$\phi(lx_1 + my_1 + n,\ lx_2 + my_2 + n, \dots) = 0.$$

Applying now Theorem I, the result follows.

For the particular case $r = 3$, it is well-known that the perpendiculars $p_1, p_2, p_3$ are proportional to the areal tangential co-ordinates of the line referred to the triangle $z_1 z_2 z_3$. We have therefore the important theorem :

THEOREM 3. *The real foci of the envelope given by the real areal tangential equation $\phi(l, m, n) = 0$, are determined by the roots of the equation $\phi(z - z_1)(z - z_2)(z - z_3) = 0$, where $z_1, z_2, z_3$ are the complex vectors to the points of reference.*

Several well-known results are seen to be cases of this theorem.

Thus :

(1) Since $\frac{k_1}{l} + \frac{k_2}{m} + \frac{k_3}{n} = 0$ is the tangential equation of an inscribed conic, it follows that, when $k_1, k_2, k_3$ are real, the roots of $(k_1/z - \alpha) + (k_2/z - \beta) + (k_3/z - \gamma) = 0$ represent the foci of an inscribed conic. In particular, the roots of $\Sigma\,(1/z - \alpha) = 0$, or $\frac{d}{dz}\{(z-\alpha)(z-\beta)(z-\gamma)\} = 0$, are the foci of the maximum inscribed ellipse.*

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* These theorems have been known for a long time; they will also be found in :

(1) Frank Morley : *Q. J. M.*, Vol. XXV, p. 186.
(2) M. Bocher : *Annals of Maths.*, Vol. VII. (1992—93), p. 70.
(3) Heawood : *Q. J. M.*, Vol. 33, p. 84.
(4) M. T. Naraniengar : this *Journal*, Vol. IV, p. 96 and Vol. V, p. 14.

(2) The foci of the conic which has $(k_1, k_2, k_3)$ for its centre, and the triangle of reference $\alpha\beta\gamma$ for a self-polar triangle, are given by the roots of the equation $k_1(z-\alpha)^2 + k_2(z-\beta)^2 + k_3(z-\gamma)^2 = 0$.

(3) The foci of the conic envelope

$$\mathrm{A}\,l^2 + \mathrm{B}m^2 + \mathrm{C}n^2 + 2\mathrm{F}mn + 2\,\mathrm{G}n\,l + 2\,\mathrm{H}\,lm = 0,$$

are given by the roots of

$$\mathrm{A}(z-\alpha)^2 + \ldots + 2\mathrm{F}(z-\beta)(z-\gamma) + \ldots = 0.$$

Since the centre is the mid-point of the foci, it easily results that its co-ordinates are $(\mathrm{A}+\mathrm{H}+\mathrm{G},\ \mathrm{H}+\mathrm{B}+\mathrm{F},\ \mathrm{G}+\mathrm{F}+\mathrm{C})$. The condition that the conic be a parabola is that one of its foci be at infinity, or

$$\mathrm{A}+\mathrm{B}+\mathrm{C}+2\mathrm{F}+2\mathrm{G}+2\mathrm{H} = 0.$$

(4) The roots of $\Sigma \tan\alpha\,(z-\alpha)^2 = 0$, coincide at the ortho-centre, being the coincident foci of the polar circle.

(5) *The focus of an inscribed parabola lies on the circum-circle.* For, one root of $\Sigma\, k_1/(z-\alpha) = 0$ is infinite, if $k_1 + k_2 + k_3 = 0$; then the other root is $z = -(\Sigma\, k_1\,\beta\gamma)/\Sigma\, k_1\alpha$, and this point lies on the circum-circle as $\Sigma\, k_1(z\alpha + \beta\gamma) = 0$.*

(6) The cubicovariant points of $\alpha, \beta, \gamma$ are the finite roots of the three equations of type $1/(z-\beta) + 1/(z-\gamma) - 2/(z-\alpha) = 0$. These points lie on the circum-circle.

## 2. Foci of curves inscribed to an m-point.

Let $(x_1, y_1), (x_2, y_2), \ldots (x_m, y_m)$ be a set of $m$ real points, $\pi_k = \mathrm{L}x_k + \mathrm{M}y_k + \mathrm{N} = 0$, $(k = 1, 2 \ldots, m)$ their tangential equations. Then the curves of class $m-1$ determined by the real equation

$$\frac{k_1}{\pi_1} + \frac{k_2}{\pi_2} + \ldots + \frac{k_m}{\pi_m} = 0, \quad \ldots \quad \ldots \quad (1)$$

* Hardy's *Pure Mathematics*, Second Edition, p. 97., Ex. 28.

constitute a family touching the $m(m-1)/2$ joins of the $m$ points. Hence by Theorem II the foci of these curves are given by the roots of the corresponding equations

$$\frac{k_1}{z-z_1}+\frac{k_2}{z-z_2}+\ldots+\frac{k_m}{z-z_m}=0,$$

where $z_k = x_k + iy_k$ is the complex vector to the $k$th point. Also if $\delta_r, \delta_s$ be the perpendiculars from the points $z_r, z_s$ to the tangent of (1) consecutive to $z_r z_s$, it is easy to see that

$$k_r\delta_s + k_s\delta_r = 0.$$

Hence the contact-point $z_{rs}$ of $z_r z_s$ with (1) is the centroid of masses $\frac{1}{k_r}, \frac{1}{k_s}$ at $z_r, z_s$. In particular, the curve

$$\frac{1}{Lx_1+My_1+N}+\frac{1}{Lx_2+My_2+N}+\ldots=0,$$

touches each of the $m(m-1)/2$ joins at its mid-point. Its foci are given by $f'(z)=0$, where $f(z)=(z-z_1)(z-z_2)\ldots(z-z_m)$.*

*Cor.* (1) If $k_1+k_2+\ldots+k_m=0$, the curve (1) has a focus at infinity, that is, touches the line at infinity. If in addition, $k_1z_1+k_2z_2+\ldots+k_mz_m=0$, the line at infinity is a bitangent, and so on.

*Cor.* (2) If $k_r+k_s=0$, $z_r z_s$ is an asymptote to the curve. Hence a curve of the type (1) can have at most $m^2/4$ or $(m^2-1)/4$ of the $m(m-1)/2$ joins, as asymptotes.

*Cor.* (3) If $\alpha, \beta, \gamma, \delta$ are points given as roots of a quartic equation $f(z)=0$, the roots of the Sextic co-variant of $f(z)$ are the roots of the three quadratic equations of the form

$$(1/z-\alpha)+(1/z-\beta)-(1/z-\gamma)-(1/z-\delta)=0.$$

The roots of this quadratic are the foci of a class cubic touching $\alpha\beta, \gamma\delta$ at their mid-points and having $\alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta$ for asymptotes, and also touching the line at infinity.

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* This theorem is given by Heawood. *Quarterly Journal of Mathematics*, Vol. 33, p. 84.

### 3. A Geometrical Interpretation of the Jacobian of two binary forms.

No general method is known for interpreting co-variants of two binary forms by geometrical constructions in the complex plane. The only general (that is, independent of the orders of the forms concerned) result hitherto obtained, is a dynamical interpretation of the Jacobian of two binary forms, due to M. Bocher.* If $f_m(z)$, $f_n(z)$ are the two forms, having the points $a_1, a_2 \ldots a_m$, $b_1, b_2 \ldots b_n$, respectively for their roots, Bocher's theorem is that the roots of the Jacobian of $f_m$, $f_n$† are the positions of equilibrium of the field of force due to attracting masses at $a_1, a_2, \ldots a_m$ each of magnitude $n$, and repelling (or negative) masses at $b_1, b_2 \ldots b_n$, each of magnitude $m$, the law of attraction or repulsion being that of the inverse distance. I offer here a purely geometric interpretation of the Jacobian.

Let $\alpha_p = 0$, $\beta_q = 0$ be the tangential equations of $a_p$, $b_q$ respectively ($p = 1, 2, \ldots m$; $q = 1, 2, \ldots n$). By theorem I, the foci of the curve

$$\frac{k_1}{\alpha_1} + \frac{k_2}{\alpha_2} + \ldots + \frac{k_m}{\alpha_m} + \frac{k_1'}{\beta_1} + \ldots + \frac{k'_n}{\beta_n} = 0 \qquad \ldots \quad (1)$$

are the roots of

$$\frac{k_1}{z - a_1} + \ldots + \frac{k_m}{z - a_m} + \frac{k'_1}{z - b_1} + \ldots + \frac{k'_n}{z - b_n} = 0 \qquad \ldots \quad (2)$$

If $k_1 = k_2 = \ldots = k_m = n$, $k'_1 = k'_2 = \ldots = k'_n = -m$, one of the roots of (2) becomes infinite, and the remaining $m + n - 2$

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* *Proceedings of the American Academy of Arts and Sciences*, Vol. 40, (1904). p. 469.

† The Jacobian of $f_m(z)$, $f_n(z)$ is

$$\begin{vmatrix} \dfrac{df_m}{dz} & \dfrac{df_m}{dt} \\ \dfrac{df_n}{dz} & \dfrac{df_n}{dt} \end{vmatrix}$$

where $t$ is a unit variable introduced for homogeneity.

roots are the roots of the Jacobian of $f_m(z)$ and $f_n(z)$. With these values of the $k$'s, (1) becomes a definite curve determined by the two polygons, $a_1a_2 \ldots a_m$, $b_1b_2 \ldots b_n$, namely, the parabola of class $m+n-1$, which touches the sides and diagonals of each of the polygons at their mid-points, and touches each cross-connector $a_rb_s$ at a point dividing it externally in the ratio $n:m$ Accordingly, the Jacobian may be interpreted as representing the foci of this curve.

For the case $m = n$ (that is, when the two forms are of equal order), the parabola (1) has all the cross-connectors for asymptotes. Writing

$$f(z) = (z-a_1)(z-a_2)\ldots(z-a_n), \quad \phi(z) = (z-b_1)(z-b_2)\ldots(z-b_n),$$

we see that polynomials of the form $f(z) + \lambda\phi(z)$ (where $\lambda$ is real) have for roots, the foci of the pencil of envelopes

$$\alpha_1\alpha_2 \ldots \alpha_n + \lambda\beta_1\beta_2 \ldots \beta_n = 0$$

These are curves of class $n$ touching all the cross-connectors.

From the algebraic significance of the Jacobian, it follows that the double-foci of envelopes of this family are roots of the equation

$$\frac{1}{z-a_1} + \frac{1}{z-a_2} + \ldots + \frac{1}{z-a_n} - \frac{1}{z-b_1} - \frac{1}{z-b_2} - \ldots - \frac{1}{z-b_n} = 0.$$

*Cor.* (1). The finite root of $(z-\alpha)(z-\beta)-(z-\gamma)(z-\delta) = 0$ is the focus of the parabola touching $\alpha\gamma$, $\alpha\delta$, $\beta\gamma$, $\beta\delta$. Thus we get an interpretation for the point $z = (\alpha\beta - \gamma\delta)/(\alpha+\beta-\gamma-\delta)$, which is of importance in connection with the complex geometry of the binary quartic.*

*Cor.* (2). The point-groups $f(z) + \lambda\phi(z) = 0$ were interpreted above as the foci of class $n$-ics touching all the cross connectors $a_rb_s$ of the point-groups $f(z) = 0$, $\phi(z) = 0$. *The locus of centroids of* $f(z) + \lambda\phi(z) = 0$ *for different real values of* $\lambda$, *is a straight line, namely, the join of the centroids of* $a_1a_2 \ldots a_n$, $b_1b_2 \ldots b_n$. This

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* Compare Russell; The Geometry of the Quartic, *Proc. Lond. Math. Soc.*, Vol. 19 (1887), p. 56.

follows immediately from the equation. The fact that the centre-locus of conics inscribed to a quadrilateral is the join of the mid-points of the diagonals, is a particular case of this theorem.

*Cor.* **(3).** If $c$ be a root of $f(z) + \lambda\phi(z) = 0$, the sum of the inclinations of $ca_1 \cdot ca_2, \ldots ca_n$ with a fixed line is equal to the sum of the inclinations of $cb_1, cb_2, \ldots cb_n$ with the same line.

*Cor* (4). The locus of the roots of $f(z) + \lambda\phi(z) = 0$ is a $(2n - 1)$-ic with $(n - 1)$-ple points at the circular points at infinity.* The degree of the locus reduces to $2n - 2$, if $a_1 a_2 \ldots a_n$, $b_1 b_2 b_n$ have the same centroid.

*Cor.* (5). The roots of $(z - a)^n + \lambda(z - b)^n = 0$ lie on the circles of the co-axal system, whose limiting points are $a$, $b$.

*Cor.* (6). Since $f'(z) = 0$ gives the foci of the class $(n - 1)$-ic touching the sides and diagonals of $a_1 a_2 \ldots a_n$ at their middle points, it follows that the roots of $f(z) + \lambda = 0$ are the vertices of an $n$-point whose sides and diagonals are touched at their mid-points by a confocal of this class $(n - 1)$-ic.

*Cor.* (7). *The roots of the first polar of* $a$ *in regard to*

$$f(z) = (z - a_1)(z - a_2)(z - a_3)$$

*may be interpreted as the finite foci of a parabola touching the six sides of* $a_1 a_2 a_3 a$.

For the first polar points are given by $\Sigma(a_1 - a)(z - a_2)(z - a_3) = 0$. This equation may be also written

$$\left(\sum \frac{1}{z - a_1}\right) - \frac{3}{z - a} = 0,$$

which shews that the sought points are the finite foci of the parabola which touches $a_1 a_2$, $a_2 a_3$, $a_3 a_1$ at their mid-points, and also touches $aa_1$, $aa_2$, $aa_3$.

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* Hilton: *Plane Algebraic Curves*, p. 72, Ex. 5.

## 4. The Miquel point of a quadrangle.

If $a_1a_2a_3a_4$ be a quadrangle, Clifford has shewn that the focus F of the unique two-fold parabola touching its six sides, has the property that the feet of the perpendiculars from F on the six sides lie on a conic F.* We may term F the 'Miquel point' of the quadrangle $a_1a_2a_3a_4$.

To find the complex co-ordinate of the Miquel point, we observe that the foci of class cubics touching the six sides of $a_1a_2a_3a_4$ are roots of an equation of the form

$$\frac{k_1}{z-a_1}+\frac{k_2}{z-a_2}+\ldots+\frac{k_4}{z-a_4} \qquad \ldots \quad (1)$$

If two roots of this equation are to be infinite, we must have

$$\left.\begin{aligned} k_1+k_2+k_3+k_4&=0\\ k_1a_1+k_2a_2+k_3a_3+k_4a_4&=0\end{aligned}\right\} \qquad \ldots \quad (2)$$

These equations shew that $k_1$, $k_2$, $k_3$, $k_4$ are proportional to the areas of $a_2a_3a_4$, $a_3a_1a_4$, $a_1a_2a_4$, $a_2a_1a_3$. Hence there is a unique bi-parabola inscribed to the quadrangle. Its focus F being the finite root of (1) is given by

$$F=\frac{\Sigma\, k_1a_2a_3a_4}{\Sigma\, k_1a_1^2}.$$

Substituting for $a_4$ and $k_4$ from (2), we obtain after simplification:

$$\frac{F-a_3}{F-a_2}=\frac{k_3}{k_2}\cdot\frac{(a_3-a_4)\,(a_3-a_1)}{(a_2-a_4)\,(a_2-a_1)} \qquad \ldots \quad (3)$$

On equating arguments, this gives

$$a_2\hat{F}a_3=a_2\hat{a}_4a_3+a_2\hat{a}_1a_3.$$

This shews that the triangular co-ordinates of F with respect to $a_1a_2a_3$ are $(\lambda+\hat{a}_1,\ \mu+\hat{a}_2,\ \nu+\hat{a}_3)$, where $(\lambda,\ \mu,\ \nu)$ are the triangular co-ordinates of $a_4$. Now the triangular co-ordinates of isogonal conjugates in regard to $a_1a_2a_3$ are connected by the relation

* Clifford: *Coll. Math. Papers*, p. 49.

$\lambda + \lambda' = \pi + \hat{a}_1$, $\mu + \mu' = \pi + \hat{a}_2$, $\nu + \nu' = \pi + \hat{a}_3$, and the triangular co-ordinates of inverse points in regard to the circle $a_1a_2a_3$ are connected by the relation $\lambda + \lambda' = 2\hat{a}_1$, $\mu + \mu' = 2\hat{a}_2$, $\nu + \nu' = 2\hat{a}_3$ * Thus the Miquel point F is the inverse in the circle $a_1a_2a_3$, of the isogonal conjugate of $a_4$ in regard to the triangle $a_1a_2a_3$. From symmetry we see therefore that by taking the isogonal conjugate of any one of the four points $a_1, a_2, a_3, a_4$ in regard to the remaining three, and inverting it in the circle through these three, we reach the same point F, namely. the Miquel point of $a_1a_2a_3a_4$.

By equating moduli in (3), we may easily deduce that

$$Fa_1 : Fa_2 : Fa_3 = \frac{k_1}{a_2a_3} . r_1 : \frac{k_2}{a_3a_1} . r_2 : \frac{k_3}{a_1a_2} . r_3,$$

where $r_1 = a_1a_4$, $r_2 = a_2a_4$, $r_3 = a_3a_4$ We may deduce from this that the distances of the Miquel point from $a_1a_2a_3a_4$ are inversely proportional to the circum-radii of the triangles $a_2a_3a_4$, $a_1a_3a_4$, $a_1a_2a_4$, $a_1a_2a_3$. From this again we can shew by using a known result,† that $Fa_4$ is equal to the diameter of the pedal circle of $a_4$ in regard to $a_1a_2a_3$.

The positions of the Miqnel point F when $a_4$ occupies special pesitions in regard to $a_1a_2a_3$. are given below.

| | $a_4$ | F |
|---|---|---|
| (1) | orthocentre of $a_1a_2a_3$ | at infinity |
| (2) | on circle $a_1a_2a_3$ | circum-centre of $a_1a_2a_3$ |
| (3) | circum-centre of $a_1a_2a_3$ | inverse of orthocentre in the circle $a_1a_2a_3$ |
| (4) | incentre | inverse of incentre |
| (5) | inner isogonal point | outer Hessian point |
| (6) | symmedian point | inverse of centroid |
| (7) | centroid | mid-point of join of Hessian points |

* Gallatly : *Modern Pure Geometry*, 2nd Edition, Chap. VII,

† Gallatly's *Modern Pure Geometry*, p. 85.

combining these results with Cliffords, we may recapitulate as follows :

The unique biparabola touching the six sides of a quadrangle is a three-cusped hypocycloid if the quadrangle is an orthocentric one, and a harmonic biparabola* if it is cyclic. Its finite focus, namely, the Miquel point F, has the property that the feet of the perpendiculars from it on the six sides of the quadrangle lie on a conic which passes through F. This conic is never a circle; it is a rectangular hyperbola if the quadrangle is cyclic.

The condition that five points $a_3 a_4\ a_5$ may admit a Miquel focus F is the existence of a triparabola (of class 4) touching their 10 connectors. For this, it must be possible to choose $k_1 k_2 \dots k_5$ so that three of the roots of $\Sigma\ k_1 / z - a_1 = 0$ are infinite, that is, so that,

$$\Sigma\ k_1 = \Sigma\ k_1 a_1 = \Sigma\ k_1 a_1{}^2 = 0.$$

Write $a_r = x_r + iy_r$, and equate real and imaginary parts to zero. Eliminating $k_1 k_2 \dots k_5$ determinantally we obtain,

$$\left| x_r{}^2 - y_r{}^2 \quad 2x_r y_r \quad x_r \quad y_r \quad 1 \right| = 0$$

$$(r = 1, 2 \dots 5).$$

This shows that the five points $a_1 a_2 a_3 a_4 a_5$ must be on a rectangular hyperbola.

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* *Vide*, my paper on the Biparabola, this *Journal* Vol. XVII. 1927.

# ON TIDAL WAVES IN AN ELLIPTIC BASIN OF VARIABLE DEPTH HAVING SMALL ECCENTRICITY.

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The object of the present paper is to solve the two-dimensiona problem of Tidal Waves in an elliptic basin which shelves gradually from the centre to the edge, the depth being the same at all points of an ellipse which is similar and similarly situated to the bounding ellipse. The case of a circular basin of variable depth where the depth at all points of a circle concentric with the bounding circle is the same, has already been solved by Lamb.* We have taken the eccentricity to be small so that the analysis may not become tedious; but it may be remarked that the two-dimensional problem in an elliptic basin having appreciable eccentricity is solvable under certain conditions, but the analysis becomes cumbrous. So far as we are aware, the problem has not been tackled before.

For free oscillations in a sheet of water of variable depth, the elevation $\zeta$, in Cartesian co-ordinates, is given by

$$\frac{\partial}{\partial x}\left(h\frac{\partial \zeta}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial \zeta}{\partial y}\right)+\frac{\sigma^2}{g}\zeta=0 \dagger$$

or $$h\left(\frac{\partial^2 \zeta}{\partial x^2}+\frac{\partial^2 \zeta}{\partial y^2}\right)+\frac{\partial h}{\partial x}\cdot\frac{\partial \zeta}{\partial x}+\frac{\partial h}{\partial y}\cdot\frac{\partial \zeta}{\partial y}+\frac{\sigma^2}{g}\zeta=0,$$

* Lamb: *Hydrodynamics* (5th Edition). Art. 193.

† Lamb: *Hydrodynamics* (5th edition). Equation 3, Art. 193.

which, in polar co-ordinates, reduces by virtue of the relations

$$\frac{\partial}{\partial x} = \cos\theta \,.\, \frac{\partial}{\partial r} - \sin\theta \,.\, \frac{\partial}{r\partial\theta},$$

$$\frac{\partial}{\partial y} = \sin\theta \,.\, \frac{\partial}{\partial r} + \cos\theta \,.\, \frac{\partial}{r\partial\theta},$$

to

$$h\left(\frac{\partial^2\zeta}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial\zeta}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2\zeta}{\partial\theta^2}\right) + \frac{\partial h}{\partial r}\cdot\frac{\partial\zeta}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial h}{\partial\theta}\cdot\frac{\partial\zeta}{\partial\theta} + \frac{\sigma^2}{g}\zeta = 0. \quad (1)$$

The equation to the bounding ellipse of the basin may be taken as

$$\frac{a^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and the equation of any ellipse similar and similarly situated with respect to the bounding ellipse and lying inside it, is

$$\frac{x^2}{\lambda a^2} + \frac{y^2}{\lambda b^2} = 1,$$

where $\lambda$ varies from 0 to 1; or, in polar co-ordinates

$$r^2 = \frac{\lambda\, a^2 b^2}{b^2\cos^2\theta + a^2\sin^2\theta}.$$

The law according to which the basin shelves gradually from the centre to the edge may be taken to be

$$h = h_0\,(1 - \lambda)$$

$$= h_0\left(1 - r^2 \,.\, \frac{a^2+b^2}{2a^2b^2} + r^2 \,.\, \frac{a^2-b^2}{2a^2b^2}\cos 2\theta\right) \quad \ldots \quad (2)$$

Let us assume the solution of (1) in the form

$$\zeta = \sum_{m=p}^{\infty}\sum_{p=o}^{\infty} A_{m,p}\cos p\theta \,.\, r^m \quad \ldots \quad (3)$$

where $m$ and $p$ are positive integers, including zero, and $m$ is either equal to or greater than $p$.

Substituting the value of $\zeta$ and $h$, as given by (3) and (2) respectively, in the equation (1), we have after simplification

$$\sum_{m=p}^{\infty} \sum_{p=0}^{\infty} h_0 \left\{ 1 + \frac{a^2-b^2}{2a^2b^3} r^2 \cos 2\theta - \frac{a^2+b^2}{2a^2b^2} r^3 \right\} \times \Big[ (m^2-p^2) .$$

$$r^{m-2} . \cos p\theta . A_{m,p} + \{ m^3 - (p+2)^2 \} r^{m-2} . \cos (p+2)\theta . A_{m,p+2}$$

$$+ \{ m^3 - (p-2)^2 \} r^{m-2} . \cos (p-2)\theta . A_{m,p-2} \Big]$$

$$+ \sum_{m=p}^{\infty} \sum_{p=0}^{\infty} h_0 \left\{ \frac{a^2-b^2}{a^2b^2} r \cos 2\theta - \frac{a^2+b^2}{a^2b^2} r \right\} \times$$

$$\{ m . r^{m-1} . \cos p\theta . A_{m,p} + m r^{m-1} \cos (p+2)\theta . A_{m,p+2}$$

$$+ m r^{m-1} \cos (p-2)\theta . A_{m,p-2} \}$$

$$+ \sum_{m=p}^{\infty} \sum_{p=0}^{\infty} r^m h_0 \frac{a^2-b^2}{a^2b^2} \sin 2\theta \times \{ p . \sin p\theta \, A_{m,p}$$

$$+ (p+2) \sin (p+2)\theta . A_{m,p+2} + (p-2) \sin (p-2)\theta . A_{m,p-2} \}$$

$$+ \frac{\sigma^2}{g} \sum_{m=p}^{\infty} \sum_{p=0}^{\infty} r^m \{ A_{m,p} . \cos p\theta + A_{m,p+2} \cos (p+2)\theta$$

$$+ A_{m,p-2} \cos (p-2)\theta \} = 0.$$

Expressing the products of sines and cosines of angles as sums, we have

$$\sum_{m=p}^{\infty} \sum_{p=0}^{\infty} \Big[ (m^2-p^2) r^{m-2} . A_{m,p} - (m^2+2m-p^2) . r^m . \frac{a^2+b^2}{2a^2b^2} . A_{m,p}$$

$$+ \frac{\sigma^2}{gh_0} r^m . A_{m,p} + \frac{a^2-b^2}{4a^2b^2} . r^m . A_{m,p+2} \{ m^2 + 2m - \overline{p+2}^2$$

$$+ 2\,\overline{p+2} \} + \frac{a^2-b^2}{4a^2b^2} . r^m . A_{m,p-2} \{ m^2 + 2m - \overline{p-2}^2 - 2\,\overline{p-2} \} \Big] \cos p\theta$$

$$+ \dots \dots = 0$$

This must be an identity and therefore equating the co-efficient of $r^{m-2}\cos p\theta$ to zero, we have

$$(m^2 - p^2)\, A_{m,p}$$

$$= \left\{ m^2 - 2m - p^2 - \frac{\sigma^2}{gh_0} \cdot \frac{2a^2b^2}{a^2+b^2} \right\} \cdot \frac{a^2+b^2}{2a^2b^2} \cdot A_{m-2,\,p}$$

$$- \left\{ m^2 - 2m - p^2 - 2p \right\} \cdot \frac{a^2-b^2}{4a^2b^2} \cdot A_{m-2,\,p+2}$$

$$- \left\{ m^2 - 2m - p^2 + 2p \right\} \cdot \frac{a^2-b^2}{4a^2b^2}\, A_{m-2,\,p-2}.$$

Writing

$$\frac{\sigma^2}{gh_0} \cdot \frac{2a^2b^2}{a^2+b^2} = n_p(n_p - 2) - p^2,$$

$$\frac{a^2+b^2}{2a^2b^2} = k^2,$$

and

$$\frac{a^2-b^2}{2a^2b^2} = k_1^{\,2}$$

the relation between the consecutive co-efficients is given by

$$\begin{aligned}(m^2 - p^2)\, A_{m,p} = {} & \{ (m - n_p)(m + n_p - 2) \}\, k^2 . A_{m-2,\,p} \\ & - \{ (m-1)^2 - (p+1)^2 \}\, k_1^{\,2} . A_{m-2,\,p+2} \\ & - \{ (m-1)^2 - (p-1)^2 \}\, k_1^{\,2}\, A_{m-2,\,p-2} \quad \ldots \quad (4)\end{aligned}$$

In this relation it must be noted that $m$ can not be less than $p$ in virtue of the asumption (3). If we put $m = p$, the relation (4) becomes

$$0 . A_{p,p} = 0 . A_{p-2,\,p-2},$$

as both $A_{p-2,\,p}$ and $A_{p-2,\,p+2}$ are non-existent. So no relation between $A_{0,0}$, $A_{2,2}$ $A_{4,4}$, etc., can be found. Similarly no relation $A_{1,1}$, $A_{3,3}$, $A_{5,5}$ ... can be obtained. Again from the form of (4), we see that no relation between $A_{0,0}$, $A_{1,1}$ is possible. Therefore $A_{0,0}$, $A_{1,1}$, $A_{2,2}$, $A_{3,3}$, etc., are all independent, and other co-efficients are to be found in terms of these. We shall take the eccentricity to be so small as to neglect its fourth and higher powers. We shall, therefore, neglect $k_1^{\,4}$ and higher powers of $k_1$ in determining the co-efficients.

The values of the co-efficients up to $A_{8,7}$ are given below, as calculated by means of the relation (4).

$$A_{1,0} = 0$$

$$A_{2,0} = - k^2 . \frac{n_0 (n_0 - 2)}{2^2} . A_{0,0}$$

$$A_{2,1} = 0$$

$$A_{3,0} = 0$$

$$A_{3,1} = - k^2 . \frac{(n_1 - 3)(n_1 + 1)}{2.4} A_{1,1}$$

$$A_{3,2} = 0$$

$$A_{4,0} = k^4 . \frac{(n_0 - 4)(n_0 - 2) n_0 (n_0 + 2)}{2^2 . 4^2} A_{0,0} - \tfrac{1}{2} k_1^2 . A_{2,2}$$

$$A_{4,1} = 0$$

$$A_{4,2} = - k^2 . \frac{(n_2 - 4)(n_2 + 2)}{2.6} A_{2,2} + \frac{2}{3} k^2 k_1^2 . \frac{n_0 (n_0 - 2)}{2^2} . A_{0,0}$$

$$A_{4,3} = 0$$

$$A_{5,0} = 0$$

$$A_{5,1} = k^4 . \frac{(n_1 - 5)(n_1 - 3)(n_1 + 1)(n_1 + 3)}{2.4.4.6} A_{1,1} - \tfrac{1}{2} k_1^2 . A_{3,3}$$

$$A_{5,2} = 0$$

$$A_{5,3} = - k^2 . \frac{(n_3 - 5)(n_3 + 3)}{2.8} A_{3,3} + \frac{3}{4} . k^2 k_1^2 . \frac{(n_1 - 3)(n_1 + 1)}{2.4} A_{1,1}$$

$$A_{5,4} = 0$$

$$A_{6,0} = - k^6 . \frac{(n_0 - 6)(n_0 - 4)(n_0 - 2) n_0 (n_0 + 2)(n_0 + 4)}{2^2 . 4^2 . 6^2} . A_{0,0}$$
$$+ \frac{1}{2} . k^2 k_1^2 \frac{(n_0 - 6)(n_0 + 4)}{6^2} A_{2,2} + \frac{2}{3} k^2 k_1^2 \frac{(n_2 - 4)(n_2 + 2)}{2.6} A_{2,2}$$

$$A_{6,1} = 0$$

$$A_{6,2} = k^4 . \frac{(n_2 - 6)(n_2 - 4)(n_2 + 2)(n_2 + 4)}{2.4.6.8} A_{2,2}$$
$$- \frac{2}{3} k^4 k_1^2 . \frac{n_0 (n_0 - 2)}{2^2} . \frac{(n_2 - 6)(n_2 + 4)}{4.8} . A_{0,0}$$
$$- \frac{1}{2} k_1^2 . A_{4,4} - \frac{3}{4} . k^4 k_1^2 . \frac{(n_0 - 4)(n_0 - 2) n_0 (n_0 + 2)}{2^2 . 4^2} . A_{0,0}$$

$$A_{6,3} = 0$$

$$A_{6,4} = -k^2 . \frac{(n_4 - 6)(n_4 + 4)}{2.10} . A_{4,4} + \frac{4}{5} . k^2 k_1^2 . \frac{(n_2 - 4)(n_2 + 2)}{2.6} . A_{2,2}$$

$$A_{6,5} = 0$$

$$A_{7,0} = 0$$

$$A_{7,1} = -k^6 . \frac{(n_1 - 7)(n_1 - 5)(n_1 - 3)(n_1 + 1)(n_1 + 3)(n_1 + 5)}{2.4.6.46.8} A_{1,1}$$

$$+ \frac{1}{2} k^2 k_1^2 . \frac{(n_1 - 7)(n_1 + 5)}{6.8} A_{3,3} + \frac{2}{3} . k^2 k_1^2 . \frac{(n_3 - 5)(n_3 + 3)}{2.8} . A_{3,3}$$

$$A_{7,2} = 0$$

$$A_{7,3} = k^4 . \frac{(n_3 - 7)(n_3 - 5)(n_3 + 3)(n_3 + 5)}{2.4.8.10} A_{3,3}$$

$$- \frac{3}{4} . k^4 k_1^2 . \frac{(n_1 - 3)(n_1 + 1)}{2.4} . \frac{(n_3 - 7)(n_3 + 5)}{4.10} . A_{1,1}$$

$$- \frac{1}{2} . k_1^2 A_{5,5} - \frac{4}{5} . k^4 k_1^2 . \frac{(n_1 - 5)(n_1 - 3)(n_1 + 1)(n_1 + 3)}{2.4.46} A_{1,1}.$$

$$A_{7,4} = 0$$

$$A_{7,5} = -k^2 . \frac{(n_5 - 7)(n_5 + 5)}{2.12} . A_{5,5} + \frac{5}{6} . k^2 k_1^2 . \frac{(n_3 - 5)(n_3 + 3)}{2.8} . A_{3,3}$$

$$A_{7,6} = 0$$

$$A_{8,0} = k^8 . \frac{(n_0 - 8)(n_0 - 6)(n_0 - 4)(n_0 + 2) n_0 (n_0 + 2)(n_0 + 4)(n_0 + 6)}{2^2 . 4^2 . 6^2 . 8^2} A_{0,0}$$

$$- \frac{1}{2} k^4 k_1^2 . \frac{(n_0 - 8)(n_0 - 6)(n_0 + 4)(n_0 + 6)}{6^2 . 8^2} A_{2,2}$$

$$- \frac{2}{3} . k^4 k_1^2 . \frac{(n_0 - 8)(n_0 + 6)}{8^2} . \frac{(n_2 - 4)(n_2 + 2)}{2.6} A_{2,2}$$

$$- \frac{3}{4} . k^4 k_1^2 . \frac{(n_2 - 6)(n_2 - 4)(n_2 + 2)(n_2 + 4)}{2.4.6.8} A_{2,2}.$$

$$A_{8,1} = 0$$

$$A_{8,2} = -k^6 . \frac{(n_2 - 8)(n_2 - 6)(n_2 - 4)(n_2 + 2)(n_2 + 4)(n_2 + 6)}{2.4.6.6.8.10} A_{2,2}$$

$$+ \frac{2}{3} . k^6 k_1^2 . \frac{n_0 (n_0 - 2)}{2^2} . \frac{(n_2 - 8)(n_2 - 6)(n_2 + 4)(n_2 + 6)}{4 . 6 . 8 . 10} . A_{0,0}$$

$$+ \frac{1}{2} k^2 k_1^2 . \frac{(n_2 - 8)(n_2 + 6)}{6 . 10} . A_{4,4}$$

$$+ \frac{3}{4} . k^6 k_1^2 . \frac{(n_2 - 8)(n_2 + 6)}{6 . 10} . \frac{(n_0 - 4)(n_0 - 2) n_0 (n_0 + 2)}{2^2 . 4^2} A_{0,0}$$

$$+ \frac{2}{3} . k^2 k_1^2 . \frac{(n_4 - 6)(n_4 + 4)}{2 . 10} . A_{4,4}$$

$$+ \frac{4}{5} . k^6 k_1^2 . \frac{(n_0 - 6)(n_0 - 4)(n_0 - 2) n_0 (n_0 + 2)(n_0 + 4)}{2^2 . 4^2 . 6^2} A_{0,0}$$

$$A_{8,3} = 0$$

$$A_{8,4} = k^4 . \frac{(n_4 - 8)(n_4 - 6)(n_4 + 4)(n_4 + 6)}{2 . 4 . 10 . 12} A_{4,4}$$

$$- \frac{4}{5} . k^4 k_1^2 . \frac{(n_4 - 8)(n_4 + 6)}{4 . 12} . \frac{(n_2 - 4)(n_2 + 2)}{2 . 6} A_{2,2}$$

$$- \frac{1}{2} k_1^2 . A_{6,6} - \frac{5}{6} k^4 k_1^2 . \frac{(n_2 - 6)(n_2 - 4)(n_2 + 2)(n_2 + 4)}{2 . 4 . 6 . 8} A_{2,2}$$

$$A_{8,5} = 0$$

$$A_{8,6} = - k^2 . \frac{(n_6 - 8)(n_6 + 6)}{2 . 14} A_{6,6} + \frac{6}{7} k^2 k_1^2 . \frac{(n_4 - 6)(n_4 + 4)}{2 . 10} A_{4,4}$$

$$A_{8,7} = 0$$

We need not find out any more co-efficients as those given above are quite sufficient to indicate the law by which they are formed.

The complete solution may be written :

$$\zeta = A_{0,0} S_0 + A_{1,1} S_1 + A_{2,2} . S_2 + \dots \dots$$

where $S_0$, $S_1$ $S_2$, etc. are all functions of $\theta$ and $r$. As $A_{0,0}$, $A_{1,1}$ $A_{2,2}$ ... are independent, each of the functions $S_0$, S, etc. gives an independent solution.

$$S_0 = \Big[ 1 - k^2 r^2 . \frac{n_0 (n_0 - 2)}{2^2} + k^4 r^4 . \frac{(n_0 - 4)(n_0 - 2) n_0 (n_0 + 2)}{2^2 . 4^2}$$

$$- k^6 r^6 . \frac{(n_0 - 6)(n_0 - 4)(n_0 - 2) n_0 (n_0 + 2)(n_0 + 4)}{2^2 . 4^2 . 6^2} + \dots \dots \Big]$$

$$+ r^2 \cos 2\theta \left[ \frac{2}{3} k^2 k_1^2 r^2 \cdot \frac{n_0(n_0-2)}{2^2} - k^4 k_1^2 r^4 \left\{ \frac{2}{3} \cdot \frac{n_0(n_0-2)}{2^2} \cdot \right.\right.$$

$$\left. \frac{(n_2-6)(n_2+4)}{4.8} + \frac{3}{4} \cdot \frac{(n_0-4)(n_0-2)n_0(n_0+2)}{2^2 . 4^2} \right\}$$

$$+ k^6 k_1^2 r^6 \left\{ \frac{2}{3} \cdot \frac{n_0(n_0-2)}{2^2} \cdot \frac{(n_2-8)(n_2-6)(n_2+4)(n_2+6)}{4.6.8.10} \right.$$

$$+ \frac{3}{4} \cdot \frac{(n_0-4)(n_0-2)n_0(n_0+2)}{2^2 . 4^2} \cdot \frac{(n_2-8)(n_2+6)}{6.10}$$

$$\left. + \frac{4}{5} \cdot \frac{(n_0-6)(n_0-4)(n_0-2)n_0(n_0+2)(n_0+4)}{2^2 . 4^2 . 6^2} \right\}$$

$$\left. - \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \right]$$

The series independent of $\theta$ in $S_0$ is the hyper-geometric series $F(\frac{1}{2} n_0, 1 - \frac{1}{2} n_0, 1, k^2 r^2)$. This series as well as the series of which $r^2 \cos 2\theta$ is the co-efficient will not converge when $k^2 r^2 >$ or $= 1$, *i.e.* when $r^2 >$ or $= \frac{2a^2b^2}{a^2+b^2}$, unless each of these series terminates. There will be values of $r$ within the elliptic lake for which $r^2 >$ or $= \frac{2a^2b^2}{a^2+b^2}$. Each of these series will terminate when $n_0$ and $n_2$ are even integers. When $n_0$ and $n_2$ are even integers, we shall write, after re-arranging the terms,

$$S_0 = \left[ 1 - k^2 r^2 \frac{n_0(n_0-2)}{2^2} + k^4 r^4 . \frac{(n_0-4)(n_0-2)n_0(n_0+2)}{2^2 . 4^2} \right.$$

$$- k^6 r^6 . \frac{(n_0-6)(n_0-4)(n_0-2)n_0(n_0+2)(n_0+4)}{2^2 . 4^2 . 6^2} + \ldots \ldots$$

$$\left. \ldots \quad \ldots \quad \ldots \quad \ldots \text{ terminates} \right]$$

$$+ r^2 \cos 2\theta \left[ \frac{2}{3} . k^2 k_1^2 r^2 . \frac{n_0(n_0-2)}{2^2} \left\{ 1 - k^2 r^2 \frac{(n_2-6)(n_2+4)}{4.8} \right.\right.$$

$$\left. + k^4 r^4 . \frac{(n_2-8)(n_2-6)(n_2+4)(n_2+6)}{4.6.8.10} \ldots \text{ terminates} \right\}$$

$$- \frac{3}{4} k^4 k_1^2 r^4 \frac{(n_0-4)(n_0-2)n_0(n_0+2)}{2^2 . 4^2}$$

# THE JOURNAL

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## NOTES AND QUESTIONS.

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# Notes and Questions.

## Perpetual Calendar Formulæ.

1. A perpetual calendar formula involves the calculation of the number of days ($\Delta$) that have elapsed since an arbitrary fixed origin up to the given DATE ($d$) of the MONTH ($m$) of the YEAR ($100\ c + y$), the corresponding WEEK-DAY ($w$) being the remainder when $\Delta$ is divided by 7, a constant depending on the origin being added to the remainder to make Sunday (the first day of the week) correspond to $w = 1$. A number of such formulæ is known, and calendars based on them are printed on card-boards or are arranged on revolving disks. Some such calendars have been published in the pages of this Journal, and the author of one of them (Mr. Hans Raj Gupta) has explained in his note on "The Perpetual Calendar Formula," (Vol. XVI, pp. *113*—*117*) the constructon of his formula. The object of the present note is to find the general formula of which Gupta's formula and all the others that I have seen, are special cases. The system considered is Gregorian, the year being considered to begin on the first of March,—as it did originally; notice the names September, October, etc. This is usual in these formulæ, and is done in order to avoid the complication created in practice by the occurrence of the extra day of the leap-year so as to affect part of the year only. The extra day is now added at the end of the year, and leap-years are of the foım $4t - 1$, though the effect of the extra day is first felt in the succeeding year $4t$. We may denote March by $m = 1$, but it is simpler to adhere to the usual numbering and denote it by $m = 3$. January and February are the 13th and the 14th months respectively of the previous year, *e.g.*, the date ordinarily denoted as 1-1-28 is, for our purposes, 1-13-27. All congruences occurring in this note are to the modulus 7.

2. The Gregorian Calendar is supposed to begin with the year 1, the previous year being 1 B.C. (or — 1), there being no year 0. We will however take the era to commence with an year 0, which may be taken to be another name for 1 B.C.

The number of days up to, but excluding, the first day of the year $100\ c + y$ is

$$\triangle_0 \equiv 365\ (100\ c + y) + 24\ c + \left[\frac{c}{4}\right] + \left[\frac{y}{4}\right]$$

$$\equiv 5c + \left[\frac{c}{4}\right] + y + \left[\frac{y}{4}\right]$$

where $[p]$ denotes the integral part of $p$. If $c = 4k + \beta$, the above reduces to

$$\Delta_0 \equiv 5\beta + y + \left[\frac{y}{4}\right],$$

(the disappearance of the $k$-term expresses the fact that four centuries are made up of an integral number of weeks). We put

$$\Delta = \Delta_0 + f(m) + d \equiv 5\beta + y + \left[\frac{y}{4}\right] + f(m) + d \quad \ldots \quad (2.1)$$

where $f(m)$ is the number of days from the beginning of the year up to, but excluding, the first day of the given month ;

$$w \equiv \Delta + w_0 \equiv 5\beta + y + \left[\frac{y}{4}\right] + f(m) + d + w_0.$$

To evaluate $w_0$, we note that 1-3-1928 was a Thursday ($w = 5$). As $\beta = 3$, and $f(3) = 0$, we have

$$5 \equiv 15 + 28 + 7 + 0 + 1 + w_0$$

or $$w_0 = 3.$$

Substituting

$$w \equiv \Delta + 3 \equiv 5\beta + y + \left[\frac{y}{4}\right] + f(m) + d + 3 \quad \ldots \quad (2.2)$$

3. We have to evaluate $f(m)$. Obviously

$$f(m) = 30(m - 3) + g(m)$$

where $g(m)$ is the number of months *earlier* than $m$, and having 31 days. The fact that February has less than 30 days does not affect $g(m)$, as February is not earlier than any month of the year. The sequence of months of 31 days is irregular ; we make it regular by introducing into the year two months of zero length, one between July and August and the other between December and January. Every alternate month has now 31 days, and if M is the number assigned on this system to the month $m$,

$$g(m) = \left[\frac{M - 2}{2}\right] = \left[\frac{M}{2}\right] - 1,$$

or $$f(m) = 30m - 91 + \left[\frac{M}{2}\right]. \quad \ldots \quad (3.1)$$

The connection between M and $m$ is

$$\left.\begin{array}{ll} M = m, & \text{for } m = 3, 4, 5, 6, 7 \\ M = m + 1, & \text{for } m = 8, 9, 10, 11, 12 \\ M = m + 2, & \text{for } m = 13, 14. \end{array}\right\} \quad \ldots \quad (3.21)$$

or, generally,

$$M = m + \left[\frac{m}{8}\right] + \left[\frac{m}{13}\right] \quad \dots \quad (3.22)$$

for all relevant values of $m$. Substituting in (3.1) and omitting completed weeks

$$f(m) \equiv 2m + \left[\frac{m + [m/8] + [m/13]}{2}\right].$$

It can be verified that we may substitute $\left[\frac{m}{9}\right]$ for $\left[\frac{m}{8}\right]$ and $\left[\frac{m}{14}\right]$ for $\left[\frac{m}{13}\right]$, without affecting $f(m)$. We obtain four values of $f(m)$,

$$f(m) \equiv 2m + \left[\frac{m + \mu_1 + \mu_2}{2}\right] \quad \dots \quad (3.3)$$

where $\mu_1 = \left[\frac{m}{8}\right]$ or $\left[\frac{m}{9}\right]$

and $\mu_2 = \left[\frac{m}{13}\right]$ or $\left[\frac{m}{14}\right]$.

A simpler form is

$$f(m) \equiv 2m + \left[\frac{m}{2}\right] + \lambda \quad \dots \quad (3.4)$$

where $\lambda = 1$ for $m = 9, 11, 13, 14$, and $= 0$ for other months. This may also be written as

$$f(m) \equiv -m + \left[\frac{7m}{2}\right] + \lambda \quad \dots \quad (3.5)$$

the advantage of this form being that $\left[\frac{7m}{2}\right]$ has only two effective values,—zero if $m$ is even, and 3 if $m$ is odd.

Another form is

$$f(m) \equiv 2m + \left[\frac{m + [m/8]}{2}\right] + \left[\frac{m}{14}\right] \quad \dots \quad (3.61)$$

$$\equiv 2m + \left[\frac{m + [m/9]}{2}\right] + \left[\frac{m}{14}\right]. \quad \dots \quad (3.62)$$

The term $\left[\frac{m}{8}\right]$ added to $m$ is an integer, but the addition of the

fractional part of $\left[\frac{m}{8}\right]$ does not affect the result, and we may write

$$f(m) \equiv 2m + \left[\frac{m + m/8}{2}\right] + \left[\frac{m}{14}\right]$$

$$\equiv 2m + \left[\frac{9m}{16}\right] + \left[\frac{m}{14}\right] \qquad \dots \quad (3.71)$$

the corresponding form for (3.62) being

$$f(m) \equiv 2m + \left[\frac{5m}{9}\right] + \left[\frac{m}{14}\right]. \qquad \dots \quad (3.72)$$

4. The question naturally arises: Can we obtain an expression of the form

$$f(m) \equiv 2m + \left[\frac{m + \phi(m)}{2}\right],$$

where $\phi(m)$ is a continuous function of $m$? In order that this may tally with one of the forms (3.3), we must have

$$M \leqslant m + \phi(m) < M + 2, \text{ for } M \text{ even}$$

and

$$M - 1 \leqslant m + \phi(m) < M + 1, \text{ for } M \text{ odd.}$$

This gives us a wide choice of $\phi$'s. We confine our attention to linear function of $m$:

$$\phi(m) = (m - a)/b,$$

$a$, $b$, being rational. The inequalities to be satisfied by $\phi(m)$ are :—

$$\left.\begin{array}{ll} \text{(i)} & -1 \leqslant \phi(m) < 1, \text{ for } m = 3 \\ \text{(ii)} & 0 \leqslant \phi(m) < 1, \text{ for } m = 4, 5, 6, 7 \\ \text{(iii)} & 0 \leqslant \phi(m) < 2, \text{ for } m = 8 \\ \text{(iv)} & 1 \leqslant \phi(m) < 2, \text{ for } m = 9, 10, 11, 12, \\ \text{(v)} & 1 \leqslant \phi(m) < 3, \text{ for } m = 13: \\ \text{(vi)} & 2 \leqslant \phi(m) < 4, \text{ for } m = 14. \end{array}\right\} \quad \dots \quad (4.11)$$

As $\phi(m)$ increases with $m$, the co-efficient of $m$ must be positive, *i.e.* $b$ is positive. As $\phi(14) > 2\,\phi(7)$, $a$ must be positive. Also (ii) involves $m \leqslant 4$, and substituting in $\phi(12)$, we get $b > 4$. Hence (i) is satisfied. If $\phi(12) < 2$, $\phi(13)$ and $\phi(14)$ are less than 3, hence (v) and part of (vi)

are satisfied. There are also other redundant conditions. Omitting these, we obtain

$$\left.\begin{aligned} 0 <\ & a \leqslant 4 \\ 7 - a <\ & b \leqslant 9 - a \\ 12 - a <\ & 2b \leqslant 14 - a \end{aligned}\right\} \qquad \ldots \quad (4.12)$$

Put $a = 2\alpha$ and $b = 7 - \alpha - \beta$; and we obtain the simple relations

$$\left.\begin{aligned} 0 < \alpha & \leqslant 2 \\ 0 \leqslant \beta & < 1 \\ \beta & < \alpha \end{aligned}\right\} \qquad \ldots \quad (4.13)$$

with

$$\phi(m) = (m - 2\alpha)/(7 - \alpha - \beta)$$

Let $\alpha = A/C$, $\beta = B/C$, where A, B, C are positive integers. Making the necessary substitutions, we obtain

$$\begin{aligned} f(m) &\equiv 2m + \left[\frac{m + \phi(m)}{2}\right] \\ &\equiv 2m + \left[\frac{(8C - A - B)\,m - 2A}{2\,(7C - A - B)}\right] \qquad \ldots \quad (4.2) \end{aligned}$$

where

$$\left.\begin{aligned} 0 < A & \leqslant 2C\,; \\ 0 \leqslant B & < C\,; \\ B & < A\,; \end{aligned}\right\}$$

as the most general expression of the required type.

5. There are only three sets of integral values of $a$ and $b$, the corresponding expressions for $f(m)$ being

$$2m + \left[\frac{6m - 3}{10}\right];$$

$$2m + \left[\frac{6m - 4}{10}\right];$$

$$2m + \left[\frac{7m - 2}{12}\right];$$

the third expression being the same as Mr. Gupta's with a change of notation.

BALAKRAM.

## A Note on the Tac-Locus.

1. The problem of finding the equation of the tac-locus of the system of curves $f(x, y, c) = 0$ without the aid of the differential equation $\phi(x, y, p) = 0$ of the system, seems to have interested some mathematicians. Workman has dwelt on this subject in two papers:

(1). On Tac-Loci: *Messenger of Mathematics*, N.S., Vol **12**.

(2). Theory of Singular Solutions, etc.: Section II, *Quarterly Journal of Mathematics*, Vol. **22**.

In the first paper, it is proved by a property of linear substitutions, that the tac-locus of

$$f(c) + c\,\phi(x, y) + \psi(x, y) = 0,$$

and of some other types of equations easily reducible to this form is included in $\dfrac{\partial(\phi, \psi)}{\partial(x, y)} = 0$. A more general result is obtained in the second paper, *viz.*

$$(\text{Tac-Discriminant})^2 = \left(\frac{p\text{-discriminant}}{c\text{-discriminant}}\right)^2$$

$$= \{\Pi\,\theta(c_1, c_2)\}^2,$$

where $c_1, c_2, \ldots c_n$ are the roots of $c$ of the primitive taken in the form $c^n + c^{n-1}\phi_1 + \ldots = 0$, $\theta$ being a complicated function of the coefficients of $c$.

Workman, in his second paper, also refers to the works of Casorati, Brioschi and Torelli, who express the tac-locus as a complicated simultaneous invariant of two quantics in $a$ and $b$ where $c = a/b$.

2. It appears to me that the following method of obtaining the equation of the tac-locus is more simple and straightforward.

*The condition that* $f(x, y, c_1) = 0$ ... ... (1)

*and* $f(x, y, c_2) = 0$ ... ... (2)

*touch at a point of intersection* $(x, y)$ *is given by*

$$\begin{vmatrix} \dfrac{\partial f(x, y, c_1)}{\partial x}, & \dfrac{\partial f(x, y, c_1)}{\partial y} \\ \dfrac{\partial f(x, y, c_2)}{\partial x}, & \dfrac{\partial f(x, y, c_2)}{\partial y} \end{vmatrix} = 0. \quad \ldots \quad (3)$$

*The tac-locus will be included in the eliminant of* $c_1$ *and* $c_2$ *between these three equations.*

NOTE 1. As the equation (3) is identically satisfied at any double-point of the curves (1) or (2), it follows that the tac-locus occurs in association with the nodal or cuspidal locus, if any, in the above eliminant,—this being the case in Workman's method also.

NOTE 2. It sometimes happens that a particular integral will figure in the above eliminant, the condition for this being that, for a particular value of $c_1$, equation (3) is satisfied solely in virtue of (1).

NOTE 3. The existence of a nodal or cuspidal locus of the system $f(x, y, c) = 0$ is an exception rather than a rule. But the existence of a tac-locus is rather a rule than an exception. We can take for $c_1$ and $c_2$ two *different* roots (or branches) defined by the equation $f(x, y, c) = 0$, and substituting in (3), the elimination can be completed.

Consider in this connection the following apparently contradictory statements:—

(i) The system $f(x, y, c) = 0$ has in general no cusp-locus.

(ii) The $p$-discriminant of $\phi(x, y, p) = 0$ in general represents the cusp-locus of the primitive. This theorem is due to Chrystal; for a proof, *vide* E. L. Ince: *Ordinary Differential Equations* (1927), pp. 83-84.

(iii) The system $f(x, y, c) = 0$ in general possesses a tac-locus.

(iv) The condition that the $p$-discriminant of $\phi(x, y, p) = 0$ might represent a tac-locus is the same as the condition that the system $\phi(x, y, c) = 0$ might possess a locus of double-points. *Vide* Ince: *loc. cit.*, p. 88.

(v) The $c$-discriminant of $f(x, y, c) = 0$ in general represents the envelope.

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These facts lead to the following conclusion:—

If $f(x, y, c) = 0$ is of the most general form (supposed to possess continuous first partial derivatives and the $c$-discriminant), the corresponding differential equation $\phi(x, y, p) = 0$ *will not be quite general,* but will be such that the $p$-discriminant includes an envelope and a tac-locus, but does not admit of a cusp-locus. Similarly, if $\phi(x, y, p) = 0$ is the most general differential equation of the first order, its primitive $f(x, y, c) = 0$ will not be quite general, but must satisfy the conditions for the possession of a cuspidal locus, while it need not possess an envelope or a tac-locus.

3. The following are a few typical examples:—

(a), $(y + c)^2 = x(x - a)(x - b)$.

The determinant (3) gives immediately

$$x = \frac{(a + b) \pm \sqrt{a^2 - ab + b^2}}{3},$$

both these straight lines being tac-loci. If however $a = b$, these lines become $x = a$, and $x = a/3$. $x = a$ is not a tac-locus but a nodal locus.

(b). $c^2 + 2cx(3a^2y^2 - 8x^2) - 3x^2a^4y^4 + a^6y^6 = 0$.

The determinant (3) on simplification becomes

$$y^3(a^2y^2 - 4x^2)^2 = 0.$$

$y = 0$ is neither a tac-locus nor a double-point locus. It is included in the particular integral corresponding to $c = 0$. $2x = \pm ay$ give the tac-loci.

(c). $c^2a^2 + c(8y^3 - 12axy) + 16ax^3 - 12x^2y^2 = 0$.

The determinant (3) gives $x = 0$, a particular integral, and $y^2 = ax$, the tac-locus.

(d). $a^3x + cxy + c^2 = 0$. The determinant (3) gives $x = 0$. This is a peculiar example in which the tac-locus is also the envelope, and at the same time, its equation gives a particular integral. When a curve $\psi(x, y) = 0$ touches every member of the system at infinity, $\psi = 0$ may be regarded as an envelope as well as a tac-locus.

C. N. SRINIVASIENGAR.

## Solutions.

### Question 156.

(J. C. SWAMINARAYAN, M.A.) :—Find completely values of $x$ and $y$ for which $x^2+y^2-1$ and $x^2-y^2-1$ are perfect squares.—

*Bhaskaracharya.*

*Solution by Profs. K. J. Sanjana and T. P. Trivedi.*

Bhaskara's statement includes two problems,* one of which is given by the proposer and the other requires values of $x$ and $y$ for which $x^2+y^2+1$ and $x^2-y^2+1$ are perfect squares.

We shall consider both the cases.

Let $$x^2+y^2\pm 1=c^2 \text{ and } x^2-y^2\pm 1=d^2;$$

$$\therefore \quad 2y^2=c^2-d^2=(c+d)(c-d).$$

Setting $$c+d=2ye \text{ and } c-d=y/e,$$

so that $$c=y\left(e+\frac{1}{2e}\right) \text{ and } d=y\left(e-\frac{1}{2e}\right),$$

we have $$c^2+d^2=y^2\left(2e^2+\frac{1}{2e^2}\right).$$

Now $$x^2\pm 1=\tfrac{1}{2}(c^2+d^2)=y^2\left(\frac{4e^4+1}{4e^2}\right).$$

Let $\frac{y}{2e}=z$, so that our equation gives

$$x^2-z^2(4e^4+1)=\mp 1.$$

Now $$\sqrt{4e^4+1}=2e^2+\frac{1}{4e^2+}\,\frac{1}{4e^2+}\,\frac{1}{4e^2+}\cdots\cdots$$

The successive convergents are

$$\frac{2e^2}{1},\ \frac{8e^4+1}{4e^2},\ \frac{32e^6+6e^2}{16e^4+1},\ \frac{128e^8+32e^4+1}{64e^6+8e^2},\ \cdots\cdots$$

---

* For further information on this problem, *vide* Dickson : *History of the Theory of Numbers*, Vol II, pp. 479-80.

The numerators of the odd convergents give the values of $x$ and the denominators those of $z$ and hence of $y$, when the upper sign is taken in both the equations. Similarly the numerators of the even convergents give the values of $x$ and the denominators those of $z$ for the equations with the lower sign.

Thus Bhaskara's problem is solved completely.

NOTE. Taking $x = 2e^2$, $y = 2e$, we find that $x = 2$, $y = 2$; $x = 8$, $y = 4$; $x = 18$, $y = 6$ are solutions of the equations with $+ 1$, when $e = 1, 2, 3$. Similarly from the third convergent taking $x = 32e^6 + 6e^2$, $y = 32e^5 + 2e$, we find that $x = 38$, $y = 34$; $x = 2072$, $y = 1028$ are solutions when $e = 1, 2$

In the same way $x = 8e^4 + 1$, $y = 8e^3$ satisfy the second set, and give $x = 9$, $y = 8$; $x = 129$, $y = 64$; $x = 649$, $y = 216$ when $e = 1, 2, 3$ So also other even convergents will give solutions for this set.

Dr. Paranjpye in his note on the solution of the equations $x^2 + y^2 - 1 = z^2$ and $x^2 - y^2 - 1 = t^2$, given on page 188, Vol. I, J.I.M.S., finds the values $x = 8p^4 + 1$, $y = 8p^3$, which agree with one pair of values in the general solution obtained above. As he himself remarks Dr. Paranjpye's solution "has in it two arbitrary assumptions," so that it is not possible to get all solutions by his method.

Prof. Sanjana has given a close translation of Bhaskara's own solution of his problem on pp. 241-42, Vol. I, J.I.M.S. Bhaskara states that we may take $y^2 = 4k^2$ and $x^2 = 5k^2 \pm 1$, or $y^2 = 36k^2$ and $x^2 = 45k^2 \pm 1$, etc. and choose $k$ so that $5k^2 \pm 1$ or $45k^2 \pm 1$ is a perfect square. This amounts to obtaining the values of $x$ and $y$ from the convergents of $\sqrt{5}$ and $\sqrt{45}$. Bhaskara adds that the problem is resolvable in many ways.

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## Question 1076.

(S. RAMANUJAM):—Prove that

(i) $\left\{ 7.20^{\frac{1}{3}} - 19 \right\}^{\frac{1}{6}} = \left(\frac{5}{3}\right)^{\frac{1}{3}} - \left(\frac{2}{3}\right)^{\frac{1}{3}}$.

(ii) $\left\{ 4.\left(\frac{2}{3}\right)^{\frac{1}{3}} - 5\left(\frac{1}{3}\right)^{\frac{1}{3}} \right\}^{\frac{1}{8}} = \left(\frac{4}{9}\right)^{\frac{1}{3}} - \left(\frac{2}{9}\right)^{\frac{1}{3}} + \left(\frac{1}{9}\right)^{\frac{1}{3}}$.

*Solution by S. Srinivasan, M.A.*

There has clearly been a mis-print as can be seen from the solution given below; the 6th root is to be taken in the first case and the 8th root in the second case.

(i) Let $(7\,.\,20^{\frac{1}{3}} - 19)^{\frac{1}{6}} = (x + y\,.\,c^{\frac{1}{3}} + zc^{\frac{2}{3}})^{\frac{1}{3}}$ where $c = 20$.

Then, $(x^2 + 2yzc) + c^{\frac{1}{3}}(cz^2 + 2xy) + c^{\frac{2}{3}}(y^2 + 2xz) = 7\,.\,20^{\frac{1}{3}} - 19.$

Equating the co-efficients of $c^{\frac{1}{3}}$, $c^{\frac{2}{3}}$, and $c^{\circ}$, we have

$$x^2 + 2yzc = -19,\ z^2c + 2xy = 7,\ y^2 + 2xz = 0.$$

Setting $x = lz$, $y = mz$, these equations become

$$m^2 = -2l,\ z^2(l^2 + 2mc) = -19,\ z^2(c + 2lm) = 7.$$

Eliminating $z$ and $l$ from the above, and simplifying

$$7m^4 - 76m^3 + 56\,.\,20m + 4\,.\,19\,.\,20 = 0.$$

On trial $m = -2$ is a solution, because

$$7\,.\,16 + 76\,.\,8 - 56\,.\,20\,.\,2 + 4\,.\,19\,.\,20 = 16\,(7 + 38 + 95 - 140) = 0$$

$$\therefore\ m = -2,\ l = -2,\ z^2(20 + 8) = 7 \text{ or } z = \pm\tfrac{1}{2}.$$

Taking $z = -\frac{1}{2}$,

$$(7\,.\,20^{\frac{1}{3}} - 19)^{\frac{1}{2}} = (1 + 20^{\frac{1}{3}} - 2^{-1}\,.\,20^{\frac{2}{3}}) = (1 + 5^{\frac{1}{3}}\,.\,2^{\frac{2}{3}} - 5^{\frac{2}{3}}\,.\,2^{\frac{1}{3}}).$$

$$= \tfrac{1}{3}(5 - 3\,.\,5^{\frac{2}{3}}\,.\,2^{\frac{1}{3}} + 3\,.\,5^{\frac{1}{3}}\,.\,2^{\frac{2}{3}} - 2) = \tfrac{1}{3}(5^{\frac{1}{3}} - 2^{\frac{1}{3}})^3$$

$$= \left\{\left(\frac{5}{3}\right)^{\frac{1}{3}} - \left(\frac{2}{3}\right)^{\frac{1}{3}}\right\}^3$$

Hence

$$\left\{7\,.\,20^{\frac{1}{3}} - 19\right\}^{\frac{1}{6}} = \left(\frac{5}{3}\right)^{\frac{1}{6}} - \left(\frac{2}{3}\right)^{\frac{1}{6}}.$$

(ii) Consider $\{4\,.\,(2)^{\frac{1}{3}} - 5\}^{\frac{1}{8}}$.

Proceeding as before let us have $(4\,.\,2^{\frac{1}{3}} - 5)^{\frac{1}{2}} = (x + y\,.\,2^{\frac{1}{3}} + z2^{\frac{2}{3}})$

$$\therefore\ x^2 + 4yz = -5;\ xy + z^2 = 2;\ y^2 + 2zx = 0.$$

If $x = lz$, $y = mz$, $m^2 = -2l$, $z^2(l^2 + 4m) = -5$, $z^2(lm + 1) = 2$.

Eliminating $l$ and $z$, $m^4 - 5m^3 + 16m + 10 = 0$.

It is evident that $m = -1$ is a root. $\therefore\ l = -\frac{1}{2}.\ z = \pm 2/\sqrt{3}$.

Taking $z = -2/\sqrt{3}$, $\{4\,.\,2^{\frac{1}{3}} - 5\}^{\frac{1}{2}} = (1 + 2\,.\,2^{\frac{1}{3}} - 2\,.\,2^{\frac{2}{3}})/3^{\frac{1}{2}}$

$$\therefore\ \{4\,.\,2^{\frac{1}{3}} - 5\}^{\frac{1}{8}} = (1 - 2\,.\,2^{\frac{1}{3}} + 2^{\frac{2}{3}})^{\frac{1}{4}}/3^{\frac{1}{8}} = (2^{\frac{1}{3}} - 1)^{\frac{1}{2}}/3^{\frac{1}{8}}$$

If $(2^{\frac{1}{3}} - 1)^{\frac{1}{2}} = (x_1 + y_1 2^{\frac{1}{3}} + z_1 2^{\frac{2}{3}})$,

$$x_1^2 + 4y_1z_1 = -1,\ x_1y_1 + z_1^2 = 0,\ y_1^2 + 2x_1z_1 = 1.$$

Setting
$$x_1 = l_1 z_1,\ y_1 = m_1 z_1,$$
$$l_1 m_1 = -1,\ z_1^2 (l_1^2 + 4m_1) = -1,\ z_1^2 (m_1^2 + 2l_1) = 1.$$
$$\therefore \quad l_1^2 + m_1^2 + 4m_1 + 2l_1 = 0.$$
Eliminating $l_1$,
$$m_1^4 + 4m_1^3 - 2m_1 + 1 = 0.$$

Evidently $m_1 = -1$ is a root. $\therefore\ l_1 = +1,\ z_1 = \pm \frac{1}{\sqrt{3}}$

Taking $z_1 = + \frac{1}{\sqrt{3}}$, we obtain
$$\left(4 . 2^{\frac{1}{3}} - 5\right)^{\frac{1}{8}} = \left(\frac{1}{3^{\frac{1}{8}}}\right)\left(\frac{1}{3^{\frac{1}{2}}}\right)\left(1 - 2^{\frac{1}{3}} + 2^{\frac{2}{3}}\right)$$
$$= \frac{1}{3^{\frac{1}{8}+\frac{1}{2}-\frac{2}{3}}}\left\{\frac{1}{9^{\frac{1}{3}}} - \left(\frac{2}{9}\right)^{\frac{1}{3}} + \left(\frac{4}{9}\right)^{\frac{1}{3}}\right\}$$
$$= \frac{1}{3^{\frac{1}{24}}}\left\{\left(\frac{1}{9}\right)^{\frac{1}{3}} - \left(\frac{2}{9}\right)^{\frac{1}{3}} + \left(\frac{4}{9}\right)^{\frac{1}{3}}\right\}$$
$$\therefore \quad 3^{\frac{1}{24}}\left\{4 . 2^{\frac{1}{3}} - 5\right\}^{\frac{1}{8}} = \left(\frac{4}{9}\right)^{\frac{1}{3}} - \left(\frac{2}{9}\right)^{\frac{1}{3}} + \left(\frac{1}{9}\right)^{\frac{1}{3}}$$
$$\text{i.e.} \quad \left\{4 . \left(\frac{2}{3}\right)^{\frac{1}{3}} - 5\left(\frac{1}{3}\right)^{\frac{1}{3}}\right\}^{\frac{1}{8}} = \left(\frac{4}{9}\right)^{\frac{1}{3}} - \left(\frac{2}{9}\right)^{\frac{1}{3}} + \left(\frac{1}{9}\right)^{\frac{1}{3}}.$$

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## Questions 1364 and 1393.

(R. GOPALASWAMI) :—Prove that a conic touching the sides of a triangle ABC at L, M and N has double contact with the polar conic of the centre of perspective of the triangles ABC, LMN ; and that the chord of contact is their axis of perspective.

Deduce that the Brocard ellipse has double contact with the circum-circle.

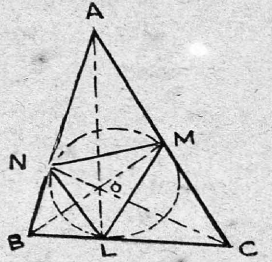
*Solutions by Ramlal Chitkara, G. V. Krishnaswami, Vidya Chandra, K. Kuppusamy and M. V. Seshadri, Partial solution by S. Audinarayanan.*

With respect to the triangle of reference

the polar line of the point $(x_0, y_0, z_0)$ is $\frac{x}{x_0} + \frac{y}{y_0} + \frac{z}{z_0} = 0,$

and the polar conic of the same point is
$$\frac{x_0}{x} + \frac{y_0}{y} + \frac{z_0}{z} = 0$$

L, M, N are the points of contact of an inconic with the sides. 'O' is the centre of perspective of the triangles ABC and LMN.




(*a*) Any conic inscribed in the triangle of reference is represented by the equation

$$\sqrt{\lambda x} + \sqrt{\mu y} + \sqrt{\nu z} = 0$$

*i.e.*, $$\Sigma\, \lambda^2 x^2 - 2\,\Sigma\, \mu\nu yz = 0.$$

The equations to the lines AL, BM, respectively are $\mu y = \nu z$ ; $\lambda x = \nu z$

Thus the point O is $\dfrac{1}{\lambda} : \dfrac{1}{\mu} : \dfrac{1}{\nu}$.

Now the polar conic of the point $\left(\dfrac{1}{\lambda} : \dfrac{1}{\mu} : \dfrac{1}{\nu}\right)$ is

$$\frac{1}{\lambda x} + \frac{1}{\mu y} + \frac{1}{\nu z} = 0, \text{ i e, } \Sigma\, \mu\nu yz = 0.$$

Again, the equation to the inconic can be written as

$$\Sigma\, \mu\nu yz - \tfrac{1}{4}\,[\Sigma\, \lambda x]^2 = 0$$

whence it follows that $\Sigma\, \mu\nu yz = 0$ has double contact with the inconic and that their chord of contact is $\Sigma\, \lambda x = 0$.

But $\Sigma\, \lambda x = 0$ is the axis of perspective of the triangles ABC and LMN, being the polar line of the centre of perspective

$$\frac{1}{\lambda} : \frac{1}{\mu} : \frac{1}{\nu}.$$

(*b*) Again, if $a, b, c$ be the lengths of the sides of the triangle of reference, the equation to the Brocard ellipse is

$$\sqrt{\frac{x}{a^2}}+\sqrt{\frac{y}{b^2}}+\sqrt{\frac{z}{c^2}}=0.$$

The centre of perspective in this case is $a^2 : b^2 : c^2$, which is the symmedian point of the triangle of reference. Its polar conic is

$$\Sigma \frac{a^2}{x}=0$$

which is the circum-circle. Hence the proposition.

Question 1393 is the same as Question 1364.

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## Question 1415.

(S. AUDINARAYANAN):—If $\sinh x \sinh y = 1$, show that

$$\int_0^\infty y\,dx=\frac{\pi^2}{4}.$$

*Solutions by S. Mahadevan, Kanwar Bahadur, S. D. Chowla, and T. Totadri Aiyangar.*

Since $\quad \sinh y=\dfrac{1}{\sinh x}$ *i.e.*, $\dfrac{e^y-e^{-y}}{2}=\dfrac{2}{e^x-e^{-x}}$

put $\quad t=e^{-x}$; then $-dx=\dfrac{dt}{t}$

and
$$\frac{e^y-1}{e^y}=\frac{4t}{1-t^2}.$$

Solving for $e^y$ we get

$$e^y=\frac{1+t}{1-t} \text{ or } -\frac{1-t}{1+t}$$

$$\therefore \qquad y=\log\frac{1+t}{1-t}$$

$$\therefore \quad \int_0^\infty y\,dx=\int_0^1\frac{dt}{t}\log\frac{1+t}{1-t}$$

$$=2\left(1^2+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right)=\frac{\pi^2}{4}.$$

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## Questions for Solution.

*Proposers of Questions are requested to send their own solutions along with their questions.*

**1515.** (N. DURAIRAJAN):—Two triangles in a plane are equilateral and concentric. Show that a three-cusped hypocycloid can be described to touch the nine mutual joins of their vertices

**1516.** (N. DURAIRAJAN):—Show that the locus of the node of a variable cubic having three fixed asymptotes and having orthogonal nodal tangents, is the radical axis of the circum-circle and the polar circle of the triangle formed by the asymptotes.

**1517.** (S. L. MALURKAR):—If the sides AB, BC, CA of a triangle be produced in order to P, Q, R, such that the perpendiculars from P on CA, Q on AB, R on BC are proportional to CA, AB, BC respectively, show that the triangles ABC, PQR are similar, and find the centre of similitude.

**1518.** (S. D. CHOWLA):—Show that

$$\cot\frac{\pi}{65} + \cot\frac{5\pi}{66} - \cot\frac{7\pi}{66} - \cot\frac{13\pi}{66} - \cot\frac{17\pi}{66} - \cot\frac{19\pi}{66}$$

$$+ \cot\frac{23\pi}{66} + \cot\frac{25\pi}{66} - \cot\frac{29\pi}{66} + \cot\frac{31\pi}{66} = 6\sqrt{11}.$$

**1519.** (R. VAIDYANATHASWAMY):—ABC is a triangle, $I, I_1, I_2, I_3$ its in and ex-centres. If $P, P'$ are isogonal conjugates, show that

$$(PI, PI_1), (PI_2, PI_3), (PA, PP'), (PB, PC)$$

are pairs of an involution.

Let K, S be the symmedian point and circum-centre of ABC, SA′ SB′, SC′ perpendiculars to the sides. Prove that

$$(SI, SI_1), (SI_2, SI_3), (SA', SK), (SB', SC')$$

are pairs of an involution.

**1520.** (M. BHIMASENA RAO) :—Show that the determinants of the third order contained in the matrix

$$\left|\begin{array}{cccc} \frac{\operatorname{sn}(2u-\theta)}{\operatorname{sn}\theta}, & \frac{\operatorname{cn}(2u-\theta)}{\operatorname{cn}\theta}, & \frac{\operatorname{dn}(2u-\theta)}{\operatorname{dn}\theta}, & 1 \\ \frac{\operatorname{sn}(2\theta-u)}{\operatorname{sn}u}, & \frac{\operatorname{cn}(2\theta-u)}{\operatorname{cn}u}, & \frac{\operatorname{dn}(2\theta-u)}{\operatorname{dn}u}, & 1 \\ 1, & 1, & 1, & 1 \end{array}\right|$$

all vanish.

**1521.** (A. NARASINGA RAO) :—If from a point on the circum-circle of a triangle lines be drawn making the same angle $\theta$ with the sides taken one way round, the corresponding points of intersection lie on a line—the Poncelet line of the point. Show that the envelope of the Poncelet lines of a triangle is a tricusp hypocycloid touching the sides.

Given any tricusp hypocycloid, show that every triangle circumscribed to it is a Poncelet triangle, *i.e*, one whose Poncelet lines touch the tricusp, and that the corresponding pedal angle $\theta$ may be found from any of the following properties :—

(1) The sides of the triangle make with any of the cusp-tangents, angles whose sum is $\theta$ (mod. $\pi$).

(2) The circum-and ortho-centres of the triangle subtend at the centre of the tricusp an angle $2\theta$.

(3) The third tangent from each vertex of the triangle makes an angle $\theta$ with its opposite side.

**1522.** (A. NARASINGA RAO) :—Show that the ortho-centre of any triangle circumscribed to a tricusp hypocycloid is the circum-centre of its residual triangle (the triangle formed by the remaining tangents from the vertices).

**1523.** (A. NARASINGA RAO) :—Show that the circum-centre of any triangle circumscribed to a tricusp hypocycloid is the ortho-centre of the triangle formed by the mid. points of the tangent chords.

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4 Annales de L'Ecole Normale Superieure, January to March 1928.
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17 Monthly Notices of the Royal Astronomical Society, **88**, 4—6
18 Philosophical Magazine, **5**, 30 and 33-34.
19 Philosophical Transactions of the Royal Society, **227**, 651-56.
20 Popular Astronomy, **36**, 4—6.

## Booklets.

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